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LETTER TO THE EDITOR

Some solutions of SU(2) Yang–Mills–Higgs system

Dipankar Ray
Department of Physics, Queen Mary College, Mile End Road, London E1 4NS, UK

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Abstract. An attempt is made to generalise the solutions of the SU(2) Yang–Mills–Higgs system obtained by Mecklenburg and O'Brien. A new soliton solution is obtained when the Higgs' field is zero.

1. Introduction

It has been shown by Mecklenburg and O'Brien (1978) that for the Lagrangian

$$\mathscr{L} = -\tfrac{1}{4}F^{\mu\nu a}F^a_{\mu\nu} - \tfrac{1}{2}\pi^{\mu a}\pi^a_\mu + \tfrac{1}{2}\mu^2(\phi^a\phi^a) - \tfrac{1}{4}\lambda(\phi^a\phi^a)^2, \qquad (1.1)$$

where $\pi^a_\mu = \partial_\mu\phi^a + e\epsilon^{abc}A^b_\mu\phi^c$ and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\epsilon^{abc}A^b_\mu A^c_\nu$, and ansatz

$$\phi^a = \hat{r}^a H(r,t)/er, \qquad A^a_0 = \hat{r}^a J(r,t)/er, \qquad A^a_i = \epsilon_{aij}\hat{r}_j(1-K(r,t))/er, \qquad (1.2)$$

where $\hat{r}^a = r^a/r$, the equations of motion reduce to

$$\begin{aligned} &r^2(H_{,rr} - H_{,tt}) = 2HK^2 + (\lambda/e^2)(H^3 - C^3r^2H), \qquad r^2J_{,rr} = 2JK^2 \\ &r^2(K_{,rr} - K_{,tt}) = K(K^2-1) + K(H^2 - J^2), \qquad rJ_{,tr} = J_{,t}, \qquad (J_{,t}\,.\,K) + 2(K_{,t}\,.\,J) = 0 \end{aligned} \qquad (1.3)$$

where $C = \mu e/\sqrt{\lambda}$ and $H_{,r} = (\partial/\partial r)H(r,t)$.

Further, they have shown that, in the limit $\lambda \to 0$ with C fixed, equations (1.3) for the time-dependent case reduce to

$$r^2(K_{,rr} - K_{,tt}) = K(K^2-1) + KH^2, \qquad r^2(H_{,rr} - H_{,tt}) = 2HK^2, \qquad J = 0. \qquad (1.4)$$

Equations (1.4) have been solved by Mecklenburg and O'Brien (1978) under the assumptions

$$K = K(y), \qquad H = H(y) \qquad (1.5)$$

where $y = y(r,t)$ and

$$r^2(K_{,rr} - K_{,tt}) = y^2K_{,yy}, \qquad r^2(H_{,rr} - H_{,tt}) = y^2H_{,yy}. \qquad (1.6)$$

In the present work we shall try instead to obtain solutions of (1.4) subject to (1.5) and $y_{,rr} - y_{,tt} = 0$, which is equivalent to

$$y = f(r+t) + g(r-t) \qquad (1.7)$$

where f and g are two functions.

The solutions obtained will include the solutions of Mecklenburg and O'Brien as a special case. Later on we shall obtain some static solutions of (1.3) in the limit $\lambda \to 0$ with C fixed as well. The reason for taking (1.7) is explained in Appendix 1.

2. Time-dependent solutions

In this section we seek solutions of (1.4) subject to (1.5) and (1.7). since throughout this section we have $J = 0$, we omit writing it here. From (1.4), (1.5) and (1.7)

$$r^2K_{,yy}(y_{,r}{}^2 - y_{,t}{}^2) = K(K^2-1) + KH^2, \qquad r^2H_{,yy}(y_{,r}{}^2 - y_{,t}{}^2) = 2HK^2. \tag{2.1}$$

Case 1. $K_{,yy} = 0 = H_{,yy}$
Here from (2.1) either

$$K = 0 \tag{2.2a}$$

or

$$H = 0, \qquad K = \pm 1. \tag{2.2b}$$

Using (1.5) and (1.7) we see that (2.2*a*) gives the following solution for (1.4):

$$K = 0, \qquad H = f(r+t) + g(r-t). \tag{2.3}$$

However, from (1.2) and (2.3) one can check that the potentials ϕ^a and A_i^a are singular at $r = 0$, and the energy integral is divergent.

Equations (1.5), (1.7) and (2.2*b*) on the other hand give a pure gauge field where the energy integral vanishes.

Case 2. K_{yy}, H_{yy} *not both zero*
Here from (2.1)

$$r^2(y_{,r}{}^2 - y_{,t}{}^2) = h(y) \tag{2.4}$$

where h is some function. From (1.7) and (2.4)

$$(u+v)^2 f_{,u}(u) g_{,v}(v) = h(y) \tag{2.5}$$

where

$$f_{,u} \equiv \mathrm{d}f(u)/\mathrm{d}u, \qquad g_{,v} \equiv \mathrm{d}g(v)/\mathrm{d}v \qquad \text{and} \qquad u = r+t, \qquad v = r-t. \tag{2.6}$$

From (2.5) and (2.6)

$$\frac{((u+v)^2 f_{,u} g_{,v})_{,u}}{((u+v)^2 f_{,u} g_{,v})_{,v}} = \frac{h_{,y} y_{,u}}{h_{,y} y_{,v}} \tag{2.7}$$

or simplifying (2.7)

$$\frac{2f_{,u}g_{,v} + (u+v)f_{,uu}g_{,v}}{2f_{,u}g_{,v} + (u+v)f_{,u}g_{,vv}} = \frac{f_{,u}}{g_{,v}}$$

or

$$2(F-G) = (u+v)(F_u - G_v) \tag{2.8}$$

where $F = 1/f_{,u}$ and $G = 1/g_{,v}$.

From (2.8) differentiating successively w.r.t. u and v

$$F_{,uu} = G_{,vv} = \tfrac{1}{2}p$$

where p is a constant, or

$$F = pu^2 + su + q, \qquad G = pv^2 + s'v + q' \tag{2.9}$$

where p, q, s, q', s' are constants. Putting (2.9) into (2.8) we get $s=0=s'$, $q'=q$, i.e.

$$f_{,u}=1/(pu^2+q), \qquad g_{,v}=1/(pv^2+q). \tag{2.10}$$

Case 2(a). $p=0$
From (1.5), (1.7) and (2.10) we note that one can without loss of generality set

$$y=r \tag{2.11}$$

Case 2(b). $q=0$
From (1.5), (1.7) and (2.10) we note that one can without loss of generality set

$$y=r/(r^2-t^2). \tag{2.12}$$

From (1.5), (2.11) and (2.12), after a little calculation, we see that for both cases 2(a) and 2(b) equations (1.6) are satisfied. Further we note that equations (2.11) and (2.12) here are respectively equations (38) and (39) of Mecklenburg and O'Brien (1978). Thus these two cases are the two cases mentioned by Mecklenburg and O'Brien (1978), and following them we note that case 2(a) is a static case and a particular solution of case 2(a) is the Prasad–Sommerfield monopole (1975). Similarly we note that a particular solution of case 2(b) is the time-dependent solution obtained by Mecklenburg and O'Brien (1978).

Case 2(c). $p\neq 0, q\neq 0, pq>0$
Putting

$$\begin{aligned} u&=\sqrt{q/p}\tan\theta_1, & -\pi/2<\theta_1<\pi/2\\ v&=\sqrt{q/p}\tan\theta_2, & -\pi/2<\theta_2<\pi/2 \end{aligned} \tag{2.13}$$

we get from (1.7), (2.6) and (2.10)

$$y=(\theta_1+\theta_2)/pq, \qquad r^2(y_{,r}{}^2-y_{,t}{}^2)=[\sin^2(\theta_1+\theta_2)]/pq. \tag{2.14}$$

Therefore equations (2.1) reduce to

$$\frac{\sin^2(y\sqrt{pq})}{pq}K_{,yy}=K(K^2-1)+KH^2, \qquad \frac{\sin^2(y\sqrt{pq})}{pq}H_{,yy}=2HK^2. \tag{2.15}$$

A particular solution of (2.15) is given by

$$H=0, \qquad K=\cos(y\sqrt{pq}) \tag{2.16}$$

where p, q are constants with $pq>0$, and y is given by (2.6), (2.13) and (2.14).

For the solution in (2.16), obviously K is finite everywhere, and from (2.13) and (2.14) we note that for K given by (2.16) as $r\to 0$, $(1-K)\sim r^2$.

Then from the explicit expression for the energy integral in terms of H, K and their derivatives given by Mecklenburg and O'Brien (1978) one can easily show that for H and K given by (2.16) the energy integral is convergent.

Case 2(d). $p\neq 0, q\neq 0, pq<0$
Here

$$y=\frac{1}{2\sqrt{-pq}}\ln\left|\frac{(u-\sqrt{-q/p})(v-\sqrt{-q/p})}{(u+\sqrt{-q/p})(v+\sqrt{-q/p})}\right| \tag{2.17}$$

where u, v are given by (2.6). Equations (2.1) reduce to

$$\begin{aligned}(-pq)^{-1}\sinh^2(y\sqrt{-pq})K_{,yy} &= K(K^2-1)+KH^2\\ (-pq)^{-1}\sinh^2(y\sqrt{-pq})H_{,yy} &= 2HK^2\end{aligned} \tag{2.18}$$

for $(pu^2+q)(pv^2+q)>0$, and

$$\begin{aligned}(pq)^{-1}\cosh^2(y\sqrt{-pq})K_{,yy} &= K(K^2-1)+KH^2\\ (pq^{-1}\cosh^2(y\sqrt{-pq})H_{,yy} &= 2HK^2\end{aligned} \tag{2.19}$$

for $(pu^2+q)(pv^2+q)<0$.

Obviously from (2.18) and (2.19) one cannot get a regular solution for the system. However, a particular solution of (2.18) is

$$H=0, \qquad K=\cosh(y\sqrt{-pq}) \tag{2.20}$$

where y is given by (2.17) and $pq<0$.

3. A static solution

In the limit $\lambda\to 0$ with C fixed a static solution of (1.3) is

$$K=1, \qquad H=J=Pr^2+Q/r \tag{3.1}$$

where P, Q are constants. The solution is obviously irregular.

4. Conclusions

Thus, summarily, the Prasad–Sommerfield solution (1975) and the Mecklenburg–O'Brien solution (1978) are particular cases of a class of solutions of (1.4) that satisfy (1.5) and (1.7). Three other such solutions are obtained here, given respectively by (2.3), (2.16) and (2.20), the last one being valid only in the region $(pu^2+q)(pv^2+q)>0$. Also in (3.1) we have obtained a static solution of (1.3) in the limit $\lambda\to 0$ with C fixed. Of the new solutions shown here the most interesting is the one given by (2.16), because that is a time-dependent solution that is regular with a finite energy integral.

Also, it can be noted that, although (1.7) was introduced as a condition in addition to (1.5), in fact (1.7) follows from (1.5) except when H and K satisfy *any one* of the following relations:

$$H=0, \qquad H=K+1, \qquad H=K-1, \qquad H=-K-1, \qquad H=-K+1. \tag{4.1a–e}$$

However, satisfaction of (4.1) does not preclude (1.7), as can be seen from (2.16).

In fact, when both (1.7) and (4.1) hold, the equations for cases 2(c), i.e. equations (2.15), reduce to

$$\begin{aligned}(\sin^2 y)K_{,yy} &= K(K^2-1) \quad \text{for (4.1}a)\\ (\sin^2 y)K_{,yy} &= 2K^2(K+1) \quad \text{for (4.1}b)\text{ and (4.1}d)\\ (\sin^2 y)K_{,yy} &= 2K^2(K-1) \quad \text{for (4.1}c)\text{ and (4.1}e).\end{aligned} \tag{4.2}$$

Similarly in cases 2(*a*) and 2(*b*) the equations reduce to

$$
\begin{aligned}
&y^2K_{,yy} = K(K^2-1) \quad \text{for } (4.1a)\\
&y^2K_{,yy} = 2K^2(K+1) \quad \text{for } (4.1b) \text{ and } (4.1d)\\
&y^2K_{,yy} = 2K^2(K-1) \quad \text{for } (4.1c) \text{ and } (4.1e).
\end{aligned} \tag{4.3}
$$

Thus the problem is reduced to a single ordinary differential equation. Although we have not obtained any solution of these equations other than the one in (2.16), the forms in (4.2) and (4.3) may lead to new solutions.

Proof that solutions of (1.4) that satisfy (1.5) must satisfy at least one of (1.7) and (4.1) is given in Appendix 1.

Appendix 1

From (1.4) and (1.5)

$$
\begin{aligned}
&r^2(K_{,y}{}^2(y_{,rr}-y_{,tt})+K_{,yy}(y_{,r}{}^2-y_{,t}{}^2)) = K(K^2-1)+KH^2\\
&r^2(H_{,y}{}^2(y_{,rr}-y_{,tt})+H_{,yy}(y_{,r}{}^2-y_{,t}{}^2)) = 2HK^2.
\end{aligned} \tag{A1.1}
$$

If H and K are linearly related then, substituting that linear relation into (A1.1), we see that at least one of (1.7) and (4.1) must hold. If, on the other hand, H and K are not linearly related, then from (A1.1) and (1.5)

$$
r^2(y_{,rr}-y_{,tt}) = \psi(y), \qquad r^2(y_{,r}^2-y_{,t}^2) = \chi(y) \tag{A1.2}
$$

where ψ and χ are some functions.

If $\chi(y)=0$, then either $y_{,r}-y_{,t}=0$ or $y_{,r}+y_{,t}=0$. In either case (1.7) is satisfied.

If $\chi(y)\neq 0$ then from (A1.2)

$$
y_{,uv}/y_{,u}y_{,v} = \psi(y)/\chi(y)
$$

or

$$
(\ln y_{,u})_{,v} = \left[\int (\psi(y)/\chi(y))\,\mathrm{d}y\right]_{,v}
$$

or

$$
y_{,u} = \lambda(u)\exp\left[\int ((\psi(y)/\chi(y))\,\mathrm{d}y\right]
$$

or

$$
\int \exp\left[-\int (\psi(y)/\chi(y))\,\mathrm{d}y\right]\mathrm{d}y = \int \lambda(u)\,\mathrm{d}u + \mu(v) \tag{A1.3}
$$

where μ is some function.

Since in (1.5) y can be replaced by an arbitrary function of y by making a transformation

$$
\int \exp\left[-\int (\psi(y)/\chi(y))\,\mathrm{d}y\right]\mathrm{d}y
$$

we see that (1.7) is satisfied by virtue of (A1.3).

References

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